

Totally Silver Graphs

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Abstract

A *totally silver coloring* of a graph G is a k -coloring of G such that for every vertex $v \in V(G)$, each color appears exactly once on $N[v]$, the closed neighborhood of v . A *totally silver graph* is a graph which admits a totally silver coloring. Totally silver coloring are directly related to other areas of graph theory such as distance coloring and domination. In this work, we present several constructive characterizations of totally silver graphs and bipartite totally silver graphs. We give several infinite families of totally silver graphs. We also give cubic totally silver graphs of girth up to 10.

Keywords: Graph coloring, distance coloring, silver coloring, domination.

1. Introduction

All graphs in this paper are finite and simple. A *totally silver coloring* of a graph G is a k -coloring of G such that for every vertex $v \in V(G)$, each color appears exactly once on $N[v]$, the closed neighborhood of v . A *totally silver graph* is a graph which admits a totally silver coloring. It is immediate from this definition that every graph admitting a totally silver coloring with k colors is $(k - 1)$ -regular. Totally silver colorings are introduced in [4]. The term totally silver is inspired by an International Mathematical Olympiad problem on silver matrices [6].

Totally silver colorings appear, under the name *perfect colorings*, in the study of file segmentations in a network in [1]. Totally silver colorings are also studied in [5] where they are called *strong colorings*. Totally silver coloring are directly translated to problems in other areas of graph theory. For instance, it is proved in [4] that a Δ -regular graph is totally silver, if and only if it is domatically full. That is, if and only its vertices can be partitioned into $\Delta + 1$ dominating sets.

Another area of graph theory related to totally silver colorings is distance coloring. The p th power of a graph G , denoted by G^p , is the graph with vertex set $V(G)$ in which two distinct vertices are adjacent, if and only if they are at distance at most p in G . A Δ -regular graph G is totally silver, if and only if G^2 is $\Delta + 1$ -colorable. Colorings of G^p are studied in the literature

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as *distance- p colorings*. In particular, upper bounds on the number of colors required by a distance- p coloring have been studied. If G is a graph with maximum degree Δ , then G^2 is $\Delta^2 + 1$ -colorable by Brooks' theorem. On the other hand, the closed neighborhood of a vertex of degree Δ induces a $\Delta + 1$ -clique in G^2 . So we have

$$\Delta + 1 \leq \chi(G^2) \leq \Delta^2 + 1. \quad (1.1)$$

By Brook's theorem, the upper bound above is tight only when G^2 is a $\Delta^2 + 1$ -clique. It is easily seen that this happens precisely when G is a Moore graph with diameter 2. Thus for every graph other than the 5-cycle, the Petersen graph, the Hoffman–Singleton graph, and the Moore graph with degree 57 and diameter 2 (if it exists), the above upper bound is improved to Δ^2 . For graphs with maximum degree 3, this bound is further improved in the following theorem of [2].

Theorem 1.1. [2] *Let G be a subcubic graph other than the Petersen graph. Then G^2 is 8-choosable, that is for every assignment of lists of size at least 8 to the vertices of G , there is a proper coloring of G^2 such that the color of each vertex is chosen from its list.*

Totally silver graphs are regular graphs for which the lower bound of equation 1.1 is tight. We study such graphs in this paper. In Section 2 we present several constructive characterizations of totally silver graphs, as well as a similar characterization of bipartite totally silver graphs. In Section 3 we give nontriviality conditions for cubic totally silver graphs. In Section 4 we present several families of cubic totally silver graphs. In Section 5 some cubic totally silver graphs of high girth are presented.

2. Properties and Characterization of totally silver graphs

Let c be a totally silver coloring of an r -regular graph G . We define a *colored 2-switch* in G to be the following operation. Let u_1v_1 and u_2v_2 be edges of G such that $c(u_1) = c(u_2)$ and $c(v_1) = c(v_2)$. Let G' be obtained from G by deleting the edges u_1v_1 and u_2v_2 , and adding the edges u_1v_2 and u_2v_1 . Since a colored 2-switch does not affect the colors appearing on each closed neighborhood, c is also a totally silver coloring of G' . The following theorem gives a classification of r -regular totally silver graphs.

Theorem 2.1. [1] *An r -regular graph is totally silver if and only if it can be obtained by a sequence of colored 2-switches from a disjoint union of $r + 1$ -cliques.*

The following is immediate from this theorem.

Corollary 2.2. *In a totally silver coloring of an r -regular graph G , all color classes have equal size. In particular, $|V(G)|$ is a multiple of $r + 1$.*

The following theorem gives a necessary condition for a graph being totally silver.

Theorem 2.3. [5] *Let r be odd. Then every r -regular totally silver graph is r -edge colorable.*

In the remainder of this section we focus on bipartite totally silver graphs and give a characterization similar to that of Theorem 2.1 for such graphs.

Lemma 2.4. *Let G be an r -regular bipartite totally silver graph with a bipartition (X, Y) , and let c be a totally silver coloring of G with color classes C_0, C_1, \dots, C_r . Then $|C_j \cap X| = |C_j \cap Y|$ and this value is independent of j . In other words, every color classes meets every partite set at the same number of vertices.*

Proof. Let $0 \leq j \leq r$. Since every $x \in X \cap C_j$ has r neighbors in Y , and since every $y \in Y \setminus C_j$ is adjacent to exactly one $x \in X \cap C_j$, we have

$$r|X \cap C_j| = |Y \setminus C_j| = |Y| - |Y \cap C_j|.$$

Rearranging the terms we obtain

$$(r-1)|X \cap C_j| = |Y| - |X \cap C_j| - |Y \cap C_j| = |Y| - |C_j|.$$

The right-hand side of this equation is independent of j since by Corollary 2.2 we have $|C_0| = |C_1| = \dots = |C_r|$. Thus $|X \cap C_j| = |X|/(r+1)$, and similarly $|Y \cap C_j| = |Y|/(r+1)$. On the other hand, since G is a regular graph, we have $|X| = |Y|$. Therefore, $|X \cap C_j| = |Y \cap C_j| = |V(G)|/(2r+2)$. \square

Corollary 2.5. *The order of every r -regular bipartite totally silver graph is a multiple of $2r+2$.*

Let G be a bipartite r -regular totally silver graph with a bipartition (X, Y) , and let c be a totally silver coloring of G . The colored 2-switch introduced above does not preserve the bipartite property of a graph, since the new edges added may create an odd cycle. We may amend this operation to resolve this issue by adding the condition that a colored 2-switch on the edges u_1v_1 and u_2v_2 with $c(u_1) = c(u_2)$ and $c(v_1) = c(v_2)$ is permitted, only when u_1 and u_2 belong to the same partite set of G . We may refer to this operation as the *bipartite colored 2-switch*. For every positive integer r we define a graph B_r to have vertex set

$$V(B_r) = \{x_0, x_1, \dots, x_r\} \cup \{y_0, y_1, \dots, y_r\},$$

and edge set

$$E(B_r) = \{x_i y_j : 0 \leq i, j \leq r \text{ and } i \neq j\}.$$

Then B_r is r -regular and bipartite, and $c : V(B_r) \rightarrow \{0, 1, \dots, r\}$ defined by $c(x_i) = c(y_i) = i$ for $i = 0, 1, \dots, r$ is a totally silver coloring of B_r . Indeed B_r is obtained from a complete bipartite graph $K_{r+1, r+1}$ by deleting the edges of a perfect matching. The graph B_2 is isomorphic to C_6 and B_3 is isomorphic to the hypercube graph Q_3 . It is immediate from Lemma 2.4 that B_r is the unique r -regular bipartite totally silver graph of order $2r+2$. The following theorem gives a characterization of bipartite totally silver graphs.

Theorem 2.6. *An r -regular bipartite graph is totally silver if and only if it can be obtained by a sequence of bipartite colored 2-switches from a disjoint union of copies of B_r .*

Proof. In this proof, a *switch* refers to a bipartite colored 2-switch as defined above. It is immediate from the definition of a switch that if G is obtained from an r -regular bipartite totally silver graph by means of repeated applications of switches, then G is r -regular, bipartite, and totally silver.

Let G be an r -regular bipartite totally silver graph with a bipartition (X, Y) . Let B be a disjoint union of copies of B_r which has the same order as G . Since a sequence of switches can be reversed in an obvious way, it suffices to prove that B is obtained from G via a sequence of switches. We proceed by induction on $|V(G)|$. If $|V(G)| = 2r + 2$, then G is isomorphic to $H = B_r$ since B_r is the only r -regular bipartite totally silver graph of order $2r + 2$. Let $|V(G)| > 2r + 2$, and let $c : V(G) \rightarrow \{0, 1, \dots, r\}$ be a totally silver coloring of G . Let $U = \{u_0, u_1, \dots, u_r\} \subset X$ and $V = \{v_0, v_1, \dots, v_r\} \subset Y$ such that $c(u_i) = c(v_i) = i$ for all $i = 0, 1, \dots, r$. Such sets U and V exist by Lemma 2.4. Let $0 \leq i, j \leq r$ with $i \neq j$. Then there are unique vertices $y \in N(u_i)$ and $x \in N(v_j)$ such that $c(y) = j$ and $c(x) = i$. If $y = v_j$ (and hence $x = u_i$), we do nothing. Otherwise, we switch the edges $u_i y$ and $v_j x$, which creates the edge $u_i v_j$. Performing this operation for all pairs of colors i and j , we obtain a graph G' in which $U \cup V$ induces a subgraph isomorphic to B_r . Since G' is r -regular, it is a disjoint union of this subgraph and a graph H . By the induction hypothesis, H is obtained by a sequence of switches from a disjoint union of copies of B_r . The sequence of switches introduced above, followed by this latter sequence, gives a sequence of switches which transforms G to B . \square

We conclude this section by some reformulations of the results of Theorem 2.1 and Theorem 2.6. Theorem 2.1 gives way to the following characterization of r -regular totally silver graphs.

Corollary 2.7. *A graph G is totally silver if and only if $V(G)$ can be partitioned to a union of $r + 1$ disjoint independent sets C_0, C_1, \dots, C_r such that for all distinct $i, j \in \{0, 1, \dots, r\}$, the subgraph induced by G on $C_i \cup C_j$ is 1-regular.*

This characterization can be viewed as a way of constructing an $r + 1$ -regular totally silver graph from any existing r -regular totally silver graph.

Corollary 2.8. *Let G be an r -regular totally silver graph and let C_0, C_1, \dots, C_r be as in Corollary 2.7. Let C_{r+1} be a set disjoint from $V(G)$ such that $|C_{r+1}| = |C_0|$. Then an $r + 1$ -regular totally silver graph H can be constructed from G by letting $V(H) = V(G) \cup C_{r+1}$, and adding a perfect matching between C_{r+1} and C_i for any $i \in \{0, 1, \dots, r\}$. Moreover, any $r + 1$ -regular totally silver graph can be constructed from a suitable r -regular totally silver graph in this way.*

A similar characterization of bipartite totally silver graphs can be formulated as a corollary of Theorem 2.6 as follows.

Corollary 2.9. *A bipartite graph G with bipartition (X, Y) is totally silver, if and only if $X = U_0 \cup U_1 \cup \dots \cup U_r$ and $Y = V_0 \cup V_1 \cup \dots \cup V_r$ where the sets U_i and V_j are pairwise disjoint, such that for any distinct $i, j \in \{0, 1, \dots, r\}$, the subgraph induced by G on $U_i \cup V_j$ is 1-regular.*

A construction similar to that of Corollary 2.8 may be derived easily from Corollary 2.9

3. Nontriviality conditions for cubic totally silver graphs

If G is a totally silver graph, then every connected component of G is totally silver. On the other hand, by Theorem 2.3, every cubic totally silver graph is bridgeless. Therefore, every cubic totally silver graph is 2-connected. We begin this section by proving that although a 2-edge cut may be present in a cubic totally silver graph, its existence leads to a trivial situation, that is a reduction to smaller cubic totally silver graphs.

Lemma 3.1. *Every cubic totally silver graph with a 2-edge cut is obtained from a disjoint union of two cubic totally silver graphs via a colored 2-switch.*

Proof. Let (S, T) be an edge cut in G with $\delta(S) = \{xy, x'y'\}$ and $x, x' \in S$. Since G is bridgeless, we have $x \neq x'$ and $y \neq y'$. Let c be a totally silver coloring of G with $c(x) = 1$ and $c(y) = 2$, and assume $c(x') \neq 4$. Let n be the number of vertices $v \in S$ with $c(v) = 4$. For $i = 1, 2, 3$, each $v \in S$ with $c(v) = 4$ has exactly one neighbor w with $c(w) = i$. On the other hand, each $w \in S$ with $c(w) = i$ has exactly one neighbor v with $c(v) = 4$. Thus $|c^{-1}(i) \cap S| = n$. A similar counting argument with the color 1 in place of 4 gives $c(x') = 2$ and $c(y') = 1$. Note that since $c(y) = c(x')$ and xy is an edge, xx' is not an edge. Thus one may apply a colored 2-switch on the edges xy and $x'y'$ to obtain two cubic totally silver graphs on vertex sets S and T . \square

By the above lemma, a nontrivial cubic totally silver graph is 3-connected. We proceed by giving similar reductions for cubic totally silver graphs with girth less than 6. Note that since the square of a 5-cycle and the square of $K_{2,3}$ are both isomorphic to a 5-clique, a cubic totally silver graph does not contain C_5 or $K_{2,3}$ as a subgraph.

Lemma 3.2. *Let G be a cubic totally silver graph which contains a 3-cycle. If $|V(G)| > 4$, then G can be reduced to a cubic totally silver graph on $|V(G)| - 4$ vertices.*

Proof. Let G be a connected cubic totally silver graph and let c be a totally silver coloring of G . Let $T = \{x_1, x_2, x_3\}$ induce a triangle in G . We may assume that $c(x_i) = i$ for $i = 1, 2, 3$. If there is $y \notin T$ with two neighbors in T , we construct a graph G' by removing x_1, x_2, x_3, y and all their edges from G and then connecting the two resulting degree 2 vertices by an edge. Then the restriction of c to $V(G')$ is a totally silver coloring of G' . Note that the degree 2 vertices in this construction are distinct since otherwise G has a 5-cycle. Suppose that each x_i

has a distinct neighbor y_i outside of T . Let z_1 and z_2 be the two neighbors of y_3 other than x_3 . Then $z_1, z_2 \notin T$, $c(y_1) = c(y_2) = c(y_3) = 4$ and $\{c(z_1), c(z_2)\} = \{1, 2\}$. By possibly renaming the vertices z_1 and z_2 , we may assume that $c(z_1) = 1$ and $c(z_2) = 2$. Let G'' be obtained from G by deleting the vertices x_1, x_2, x_3, y_3 and all their edges, and adding the edges y_1z_1 and y_2z_2 . The restriction of c to $V(G'')$ is a totally silver coloring of G'' . \square

Lemma 3.3. *Let G be a triangle-free cubic totally silver graph which contains a 4-cycle. If $|V(G)| > 4$, then G can be reduced to a cubic totally silver graph on $|V(G)| - 4$ vertices.*

Proof. Let G be a connected triangle-free cubic totally silver graph and let c be a totally silver coloring of G . Let $C = x_1x_2x_3x_4x_1$ be a 4-cycle in G . Each x_i has a distinct neighbor $y_i \notin C$ since G is triangle- and $K_{2,3}$ -free. By possibly renaming the colors, we may assume that $c(x_i) = i$ for $i = 1, 2, 3, 4$. Then $c(y_1) = 3$, $c(y_2) = 4$, $c(y_3) = 1$, and $c(y_4) = 2$. Let G' be obtained from G by deleting the vertices x_1, x_2, x_3, x_4 and all their edges, then adding the edges y_1y_3 and y_2y_4 . Then the restriction of c to $V(G')$ is a totally silver coloring of G' . \square

In light of the reductions of this section, we consider a cubic totally silver graph *nontrivial*, if it is 3-connected and it has girth at least 6.

4. Constructions of cubic totally silver graphs

In this section we present several infinite families of nontrivial cubic totally silver graphs. In particular, we give nontrivial cubic totally silver graphs of any order which is a multiple of 4. Note that the $(3, 6)$ -cage (the smallest cubic graph with girth at least 6) has order 14. Therefore, no nontrivial cubic totally silver graphs of orders 4, 8, or 12 exist.

As our first example, we consider the family of generalized Petersen graphs $P(n, d)$ where n and d are positive integers with $n \geq 2d + 1$. The graph $P(n, d)$ consists of an n -cycle $x_1x_2 \cdots x_nx_1$, along with n other vertices y_1, y_2, \dots, y_n where each y_i is adjacent to x_i , y_{i-d} and y_{i+d} (all subscripts are reduced modulo n). In particular, $P(4, 1)$ is the hypercube graph Q_3 , $P(5, 2)$ is the Petersen graph, and $P(n, 1)$ is the prism $C_n \square K_2$, where \square denotes the Cartesian product. The following is the main result of [3].

Theorem 4.1. [3] *The generalized Petersen graph $P(n, d)$ is totally silver if and only if $4|n$ and d is odd.*

The above theorem gives examples of cubic totally silver graphs whose order is divisible by 8. While these graphs are all 3-connected and bipartite (thus triangle-free), not all of them are C_4 -free. The graphs $P(2d + 2, d)$ where $d \geq 3$ is odd, are all nontrivial cubic totally silver graphs.

Theorem 4.2. *For all $n \geq 4$, there exists a nontrivial cubic totally silver graph of order $4n$.*

Proof. We give two families E_n and M_n of cubic graphs of order $4n$. The graph E_n is totally silver when $3|n$ and M_n is totally silver when $3 \nmid n$. Given $n \geq 3$, the graph E_n consists of three disjoint n -cycles $u_1u_2 \cdots u_nu_1$, $v_1v_2 \cdots v_nv_1$, and $w_1w_2 \cdots w_nw_1$, and n vertices z_1, z_2, \dots, z_n , where z_i is joined to u_i , v_i and w_i for each $i = 1, 2, \dots, n$. The graph E_6 is illustrated in Figure 1. It is straightforward to see that E_n is 3-connected, and that for $n \geq 6$, E_n has no 3-, 4-, or 5-cycles. If $3|n$, let $c : V(E_n) \rightarrow \{0, 1, 2, 3\}$ be defined by $c(u_i) = i \bmod 3$, $c(v_i) = i + 1 \bmod 3$, $c(w_i) = i + 2 \bmod 3$, and $c(z_i) = 3$. Then c is a totally silver coloring of E_n .

Given $n \geq 3$, the graph M_n consists of a $3n$ -cycle $u_1u_2 \cdots u_{3n}u_1$, and n vertices z_1, z_2, \dots, z_n , and edges z_iu_i , z_iu_{n+i} , and z_iu_{2n+i} for all $i = 1, 2, \dots, n$. The graph M_4 is illustrated in Figure 1. It is straightforward to check that M_n is 3-connected and that for $n \geq 4$, M_n has girth 6. If $3 \nmid n$, we define $c : V(M_n) \rightarrow \{0, 1, 2, 3\}$ by $c(u_i) = i \bmod 3$ and $c(z_i) = 3$. It is easy to verify that c is a totally silver coloring. \square

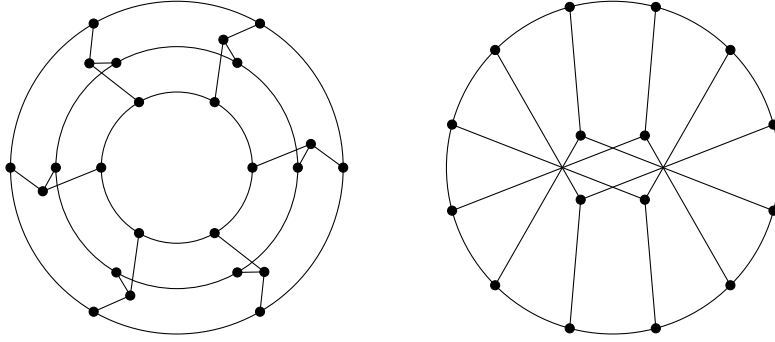


Figure 1: The graphs E_6 (left) and M_4 (right).

Remark 4.3. Indeed the graph E_n is totally silver if and only if $3|n$. This is since it is easily observed that in every totally silver coloring of E_n , all the vertices z_n receive the same color. This implies that each of the remaining n -cycles is colored by three colors. On the other hand, a 3-coloring of C_n^2 exists if and only if $3|n$. A brute-force computer search confirms that $\chi(E_5^2) = 6$. For a 5-coloring of E_n^2 where $n > 5$, one may take a 4-coloring of C_n^2 for the cycle on the u_i , the same coloring rotated one position for the cycle on the v_i , and rotated two positions for the cycle on the w_i . A fifth color is used for the z_i .

Remark 4.4. Similarly, it is easily seen that in every totally silver coloring of M_n , all z_i receive the same color, thus only three colors are available for the remaining $3n$ -cycle. On the other hand, the only 3-coloring of the square of a $3n$ -cycle has color classes which contain every third vertex on the cycle. If $3|n$, this coloring is not valid as a total silver coloring of M_n . This is since if $3|n$, then $c(x_1) = c(x_n)$, while the vertices x_1 and x_n have a common neighbor z_1 . A brute-force computer search confirms that $\chi(M_3^2) = 6$. For a 5-coloring of M_n^2 when $n > 3$, one may use four colors for the $3n$ -cycle on the x_i and a fifth color for the z_i . We omit a description of the coloring of the cycle since it involves several cases.

For $n \geq 4$, the graph L_{2n} consists of a $2n$ -cycle $x_1 \cdots x_{2n}$ and the edges $x_i x_{i+5}$ for all odd i (subscripts are reduced modulo $2n$). In particular, L_8 is isomorphic to the hypercube Q_3 and L_{14} is isomorphic to the Heawood graph. The graph L_{24} is illustrated in Figure 2. The graph L_{2n} is bipartite for all $n \geq 4$, since every edge joins an odd-numbered vertex to an even-numbered vertex. Thus by Corollary 2.5, L_{2n} is not totally silver when $4 \nmid n$.

On the other hand, L_{2n} is totally silver when $4 \mid n$. A totally silver coloring of L_{2n} in this case is more easily presented using an alternate representation of this graph. For $m \geq 1$, we construct a graph L'_m on $8m$ vertices as follows. We begin by a $6m$ -cycle $y_1 y_2 \cdots y_{6m} y_1$, and for each $1 \leq i \leq 6m$ with $i \equiv 1, 2 \pmod{6}$, we join a new vertex to the vertices y_i , y_{i+4} , and y_{i+8} (all subscripts are reduced modulo $6n$). We refer to the $2m$ new vertices added in this last step as z_1, z_2, \dots, z_{2m} . A totally silver coloring of L'_m is the map $c : V(L'_m) \rightarrow \{0, 1, 2, 3\}$ defined with $c(y_i) = i \pmod{3}$ and $c(z_i) = 3$. It remains to prove that L'_m is isomorphic to L_{8m} . We omit a formal proof of this result, and instead, we present a graphical proof in Figure 2, where in the normal drawing of the graph L_{24} (left) we highlight an 18-cycle corresponding to the main cycle of the graph L'_3 (right). The clear pattern visible in this figure may be used to give an isomorphism between the graphs L'_m and L_{8m} .

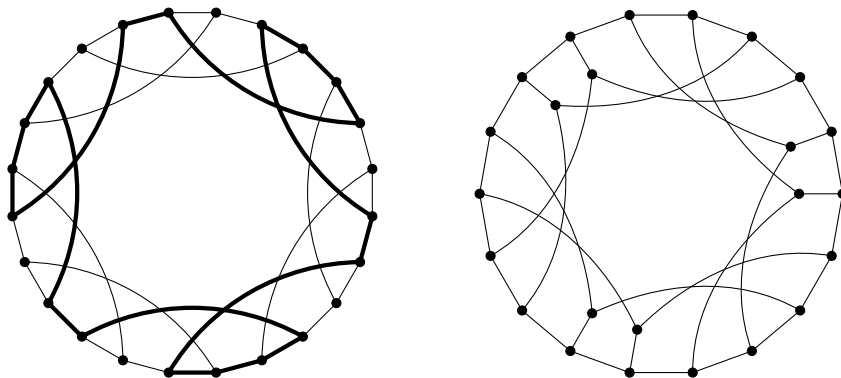


Figure 2: Two drawings of the graph L_{24} .

For a positive integer $n \geq 2$, the *Möbius ladder* V_{2n} consists of a cycle $C = v_1 v_2 \cdots v_{2n} v_1$ and the chords $v_i v_{n+i}$ for all $i = 1, 2, \dots, n$. Note that V_4 is isomorphic to K_4 and V_6 is isomorphic to $K_{3,3}$. The graph V_{2n} contains a 4-cycle $v_1 v_2 v_{n+2} v_{n+1} v_1$. Using the reduction of Lemma 3.3, and the fact that V_4 is totally silver, one can see that V_{2n} is totally silver if and only if $n \equiv 2 \pmod{4}$. We may use Möbius ladders in construction of nontrivial cubic totally silver graphs as follows. Let $n \geq 2$ and let H be obtained from V_{2n} by subdividing all edges of C . We construct a graph D_n from the disjoint union of two copies of H by adding edges between pairs of corresponding vertices of degree 2 in the two copies of H . The graph D_3 is illustrated in Figure 3.

Let $x_1 x_2 \cdots x_{4n} x_1$ and $y_1 y_2 \cdots y_{4n} y_1$ be the cycles in D_n obtained by subdividing the main $2n$ -cycle of each copy of V_{2n} used in its construction. We may shift the labels so that the edges

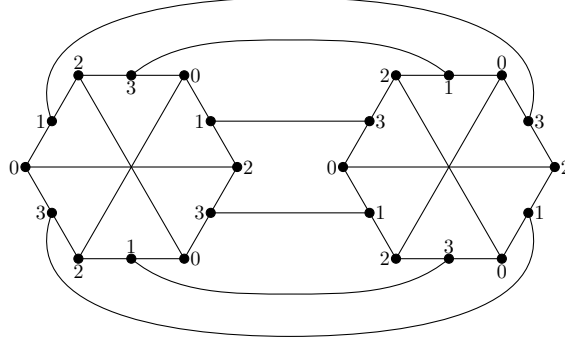


Figure 3: The graph D_3 and a totally silver coloring of it.

$x_i y_i$ are present in D_n for all even $1 \leq i \leq 4n$. If n is odd, $c : V(D_n) \rightarrow \{0, 1, 2, 3\}$ defined by $c(x_i) = i \bmod 4$ and $c(y_i) = i + 2 \bmod 4$ is a totally silver coloring. A totally silver coloring of D_3 is presented in Figure 3. Indeed the graph D_n is not totally silver if n is even. We omit the proof of this assertion.

5. Totally silver graphs with large girth

By way of Corollary 2.8, any cubic totally silver graph G is obtained from a 2-regular totally silver graph (a disjoint union of cycles if lengths divisible by 3). Here we give a construction of a family of cubic totally silver graphs with girth 9 starting from a $3n$ -cycle $C = x_1 x_2 \cdots x_{3n} x_1$ and n isolated vertices z_1, \dots, z_n . Suppose that $X = \{x_1, x_2, \dots, x_{3n}\}$ is partitioned into $X = X_1 \cup \dots \cup X_n$, where $|X_i| = 3$ for all i , and no X_i contains two vertices x_j and x_k where j and k are congruent modulo 3. Let G be the cubic graph constructed from the $3n$ -cycle C and the vertices z_1, \dots, z_n , by joining z_i to X_i , for all $i = 1, 2, \dots, n$. Then G is totally silver. The challenge here is to partition X in such a way that the resulting cubic totally silver graph G contains no short cycles. In the following we show that girth 9 can be achieved when $n \geq 15$, by taking $X_i = \{x_{3i}, x_{3i+7}, x_{3i+20}\}$ for all $i = 1, 2, \dots, n$.

Theorem 5.1. *For every $n \geq 15$, there exists a cubic totally silver graph of order $4n$ and girth 9.*

Proof. Let $n \geq 15$ and let G be the graph obtained from a $3n$ -cycle $x_0 x_1, \dots, x_{3n-1} x_0$ and n isolated vertices z_0, z_1, \dots, z_{n-1} according to the above construction, with the choice of $X_i = \{x_{3i}, x_{3i+7}, x_{3i+20}\}$ for all $i = 0, 1, \dots, n-1$. To show that G contains no cycles shorter than 9, by symmetries of the construction of G , we may only verify that none of the vertices x_0, x_1, x_{3n-1} , and z_0 is contained in a short cycle. Although cumbersome, this can be done by listing all vertices at distance at most 4 from each of these vertices. In Figure 4, we present the subgraph of G induced on the set of vertices at distance at most 4 from x_0 . This illustration affirms that there is no cycle in G of length 8 or less through the vertex x_0 . Similar illustrations

for x_1 , x_{3n-1} , and z_0 show the same conclusion for these vertices. We omit these, but note that these three subgraphs of G have significant overlap with the one illustrated in Figure 4. \square

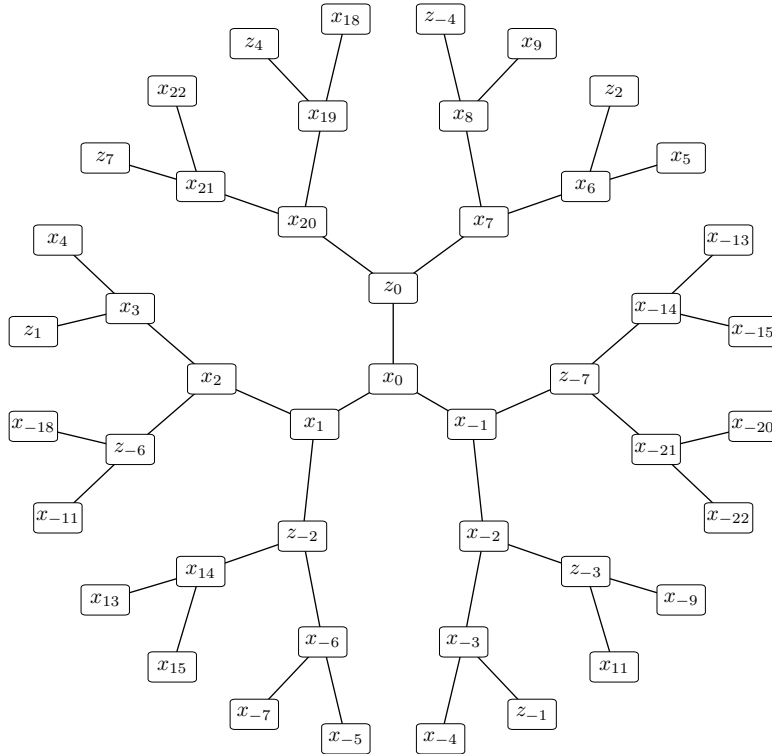


Figure 4: The subgraph induced by vertices at distance at most 4 from x_0 , in the construction of Theorem 5.1. Indices of the x_i are reduced modulo $3n$ and indices of the z_i are reduced modulo n . For better visibility, the edges x_4x_5 , x_4z_{-1} , $x_{-4}x_{-5}$, $x_{-5}z_{-4}$, z_2x_{13} , and z_7x_{-22} are omitted in this drawing.

Remark 5.2. It can be verified similarly that the above construction with $n \geq 22$ and $X_i = \{x_{3i}, x_{3i+8}, x_{3i+22}\}$ for all i , gives cubic totally silver cubic graphs of girth 10. An alternate way of obtaining (bipartite) totally silver cubic graphs of girth 10 is discussed in the remainder of this section.

Remark 5.3. The construction of Theorem 5.1, where the neighborhoods of the vertices z_i are congruent modulo 3, does not give graphs of girth higher than 10. This is since if z_0 is adjacent to x_0 and x_p , then z_1 is adjacent to x_3 and x_{p+3} , thus creating a 10-cycle $z_0x_0x_1x_2x_3z_1x_{p+3}x_{p+2}x_{p+1}x_px_0$.

Given any graph G , the *bipartite double cover* of G is the tensor product $H = G \times K_2$. Namely, H has vertex set $V(H) = V(G) \times \{1, 2\}$, and $(u, i), (v, j) \in V(H)$ are adjacent in H if and only if $uv \in E(G)$ and $i \neq j$. For example $K_{n,n}$ is the bipartite double cover of K_n . If c is a totally silver coloring of a graph G , then the map c' defined by $c'(u, i) = c(u)$ is a totally silver coloring of $G \times K_2$. It is easy to see that every m -cycle in G is mapped to a $2m$ -cycle in $G \times K_2$ if m is odd, and to two m -cycles in $G \times K_2$ if m is even. Therefore, the girth of

$G \times K_2$ is greater than or equal to the girth of G . We thus obtain (bipartite) cubic totally silver graphs of girth 10 by taking the bipartite double cover of the graphs of Theorem 5.1.

We conclude this paper by the following problem.

Problem 5.4. Do there exist cubic totally silver graphs of arbitrary high girth?

References

- [1] E. M. Bakker, J. van Leeuwen, and R. B. Tan. Perfect colorings. Technical Report RUU-CS-90-35, Department of Computer Science, Utrecht University, 1990.
- [2] D. W. Cranston and S.-J. Kim. List-coloring the square of a subcubic graph. *Journal of Graph Theory*, 57:65–87, 2008.
- [3] J. Ebrahimi, N. Jahanbakht, and E. S. Mahmoodian. Vertex domination of generalized Petersen graphs. *Discrete Mathematics*, 309(13):4355–4361, 2009.
- [4] M. Ghebleh, L. A. Goddyn, E. S. Mahmoodian, and M. Verdian-Rizi. Silver cubes. *Graphs Comb.*, 24(5):429–442, 2008.
- [5] G. Kant and J. van Leeuwen. Strong colorings of graphs. Technical Report RUU-CS-90-16, Department of Computer Science, Utrecht University, 1990.
- [6] M. Mahdian and E. S. Mahmoodian. The roots of an IMO97 problem. *Bull. Inst. Combin. Appl.*, 28:48–54, 2000.